Lecture 15
14.5 Properties of the gradient
14.6 Tangent planes

Jeremiah Southwick

February 25, 2019

Things to note

Upcoming dates:
Wednesday, February 27: Quiz 7
Monday, March 4: Quiz 8 and WF drop date (see grade calculation sheet on Blackboard)
Wednesday, March 6: Review
Friday, March 8: Exam 2

Last class

Definition

The gradient of $f(x, y)$ *is*

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$(D_{\vec{\mathbf{u}}} f)(a, b) = [\nabla f(a, b)] \cdot \vec{\mathbf{u}}.$$

Geometric info about gradient

Consider $(D_{\vec{\mathbf{u}}} f)(a, b) = [\nabla f(a, b)] \cdot \vec{\mathbf{u}}$.

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Question

What's another formula for $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$*?*

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Answer

$||\vec{\mathbf{v}}||||\vec{\mathbf{w}}|| \cos(\theta)$.

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Question

What's another formula for $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$?

Answer

$\|\vec{\mathbf{v}}\| \|\vec{\mathbf{w}}\| \cos(\theta)$.

Thus

$$(D_{\vec{\mathbf{u}}} f)(a, b) = [\nabla f(a, b)] \cdot \vec{\mathbf{u}} = \|\nabla f(a, b)\| \|\vec{\mathbf{u}}\| \cos(\theta) = \|\nabla f(a, b)\| \cos(\theta)$$

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$$(D_{\vec{\mathbf{u}}}f)(a,b) = \|\nabla f(a,b)\| \cos(\theta)$$

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Which θ maximize the directional derivative? Which θ minimize it? Which θ make it 0?

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Which θ maximize the directional derivative? Which θ minimize it? Which θ make it 0?

At (a,b), f increases most rapidly in the direction of $\nabla f(a,b)$. $(\theta = 0)$.

At (a,b), f decreases most rapidly in the opposite direction of $\nabla f(a,b)$. $(\theta = \pi)$.

At (a,b), f is neither rising nor dropping in the directions orthogonal to $\nabla f(a,b)$. $(\theta = \pi/2)$.

Gradient example

Example

Let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + xy - x$. *Find the directions in which*

(a) f *increases most rapidly at* $(1,1)$.

(b) f *decreases most rapidly at* $(1,1)$.

(c) the height of f *does not change (i.e., slope is 0).*

Gradient example

Example

Let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + xy - x$*. Find the directions in which*

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(c) the height of f *does not change (i.e., slope is 0).*

We have $\nabla f = \langle x + y - 1, y + x \rangle$. Thus $\nabla f(1,1) = \langle 1, 2 \rangle$.
The direction of most rapid increase will be the unit vector in the direction of $\langle 1, 2 \rangle$, or $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$.
The direction of most rapid decrease will be the unit vector in the opposite direction of $\nabla f(1,1)$, or $\left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$
The direction of zero change is any direction orthogonal to the gradient at $(1,1)$. These two vectors are $\left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$ and $\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$.

Properties of the gradient (page 836)

The gradient is essentially a derivative vector, so it satisfies many properties of derivatives.

1,2. $\nabla(f \pm g) = \nabla f \pm \nabla g$

3. $\nabla(kf) = k\nabla f$ for any number k

4. $\nabla(fg) = f\nabla g + g\nabla f$

5. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

14.6 Tangent planes

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How do we find a plane that is tangent to a function $z = f(x, y)$*?*

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How do we find a plane that is tangent to a function $z = f(x, y)$?

First we need a normal vector $\vec{\mathbf{n}}$ and a point $P_0 = (a, b, c)$ in the plane. Then the equation is

$$\vec{\mathbf{n}} \cdot \langle x - a, y - b, z - c \rangle = 0.$$

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Idea: First find some lines tangent to the function.

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Start off with tangent lines in the x- and y-directions.

Tangent planes

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How do we find the slope of a function $z = f(x, y)$ *in the* x*-direction?*

Tangent planes

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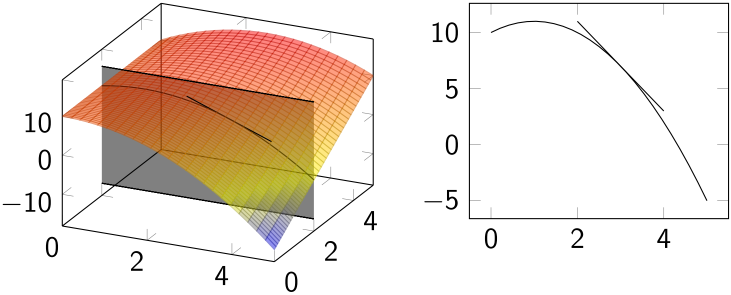

Tangent planes

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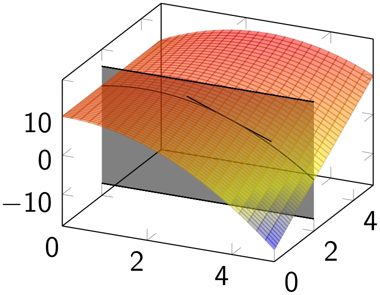
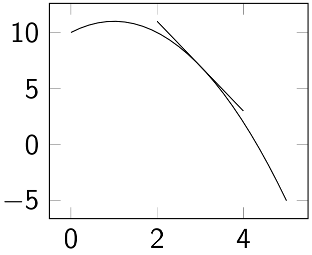


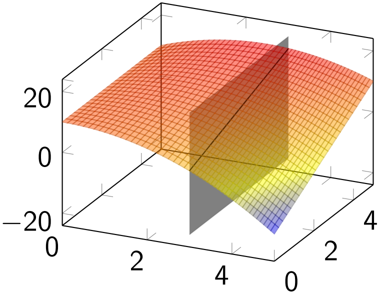


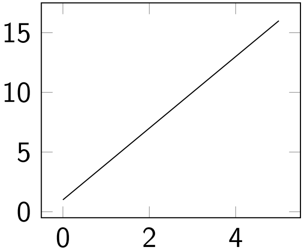
This line is given by $z - f(a, b) = f_x(a, b)(x - a)$ or

$$\vec{\mathbf{r}}(t) = \langle a, b, f(a, b)\rangle + t\langle 1, 0, f_x(a, b)\rangle.$$

Tangent planes

We can do the same thing in the y-direction.





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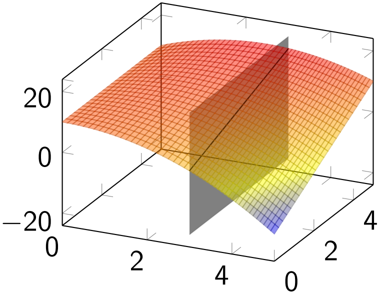

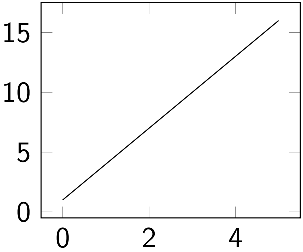

This line is given by $z - f(a, b) = f_y(a, b)(y - b)$ or

$$\vec{\mathbf{r}}(t) = \langle a, b, f(a, b)\rangle + t\langle 0, 1, f_y(a, b)\rangle.$$

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Tangent planes

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Two tangent lines:

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and

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Point: $P_0 = (a, b, f(a, b))$

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Point: $P_0 = (a, b, f(a, b))$
Normal vector: $\vec{\mathbf{n}}$ is the cross product of the direction vectors of the lines.

Tangent planes

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$$\vec{\mathbf{n}} =$$

Tangent planes

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$$\vec{\mathbf{n}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 1 & f_y(a, b) \\ 1 & 0 & f_x(a, b) \end{vmatrix} = \langle f_x(a, b), f_y(a, b), -1 \rangle$$

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Then the equation of the plane is $\vec{\mathbf{n}} \cdot \langle x - a, y - b, z - f(a, b) \rangle = 0$ or

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0.$$

Solving for z, we have

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Tangent plane example

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Example

Find the tangent plane to $z = x\cos(y) - ye^x$ *at* $(\ln(2), 0, \ln(2))$.

Tangent plane example

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

Example

Find the tangent plane to $z = x\cos(y) - ye^x$ *at* $(\ln(2), 0, \ln(2))$*.*

We have $\nabla f = \langle \cos(y) - ye^x, -x\sin(y) - e^x \rangle$ and thus the equation is

$$(x - \ln(2)) + (-2)(y - 0) - (z - \ln(2)) = 0$$

or

$$z = x - 2y.$$